

On Generating Gravity Waves with Matter and Electromagnetic Waves

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Abstract

If a homogeneous plane light-like shell collides head-on with a homogeneous plane electromagnetic shock wave having a step-function profile then no backscattered gravitational waves are produced. We demonstrate, by explicit calculation, that if the matter is accompanied by a homogeneous plane electromagnetic shock wave with a step-function profile then backscattered gravitational waves appear after the collision.

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1 Introduction

This paper is concerned with the production of gravitational waves from the interaction of matter with light in the context of Einstein–Maxwell classical field theory. We present an explicit physical mechanism for the generation of gravitational waves by such a process.

In general matter interacting with light does not produce gravitational waves. A simple illustration of this is the head–on collision of a homogeneous plane light–like shell with a homogeneous plane electromagnetic shock wave having a profile described by the Heaviside step–function (see [1] (eqs.(3.9)–(3.11)) where the electromagnetic shock wave is also accompanied by a homogeneous plane light–like shell with no resulting backscattered gravitational waves after the collision). This solution of the vacuum Einstein–Maxwell field equations fits the pattern of all known post–collision space–times (see for example [2], [3]): The line–element of the space–time after the head–on collision of homogeneous, linearly polarized, plane light–like signals has the Rosen–Szekeres form [4][5]

$$ds^2 = -e^{-U} (e^V dx^2 + e^{-V} dy^2) + 2 e^{-M} du dv , \quad (1.1)$$

with U, V, M functions of (u, v) . The Maxwell field has only two (real–valued) Newman–Penrose components ϕ_0 and ϕ_2 which are both functions of (u, v) . Writing $\hat{\phi}_0 = e^{-U/2} \phi_0$ and $\hat{\phi}_2 = e^{-U/2} \phi_2$, the vacuum Maxwell field equations read

$$\frac{\partial \hat{\phi}_0}{\partial u} = -\frac{1}{2} V_v \hat{\phi}_2 , \quad \frac{\partial \hat{\phi}_2}{\partial v} = -\frac{1}{2} V_u \hat{\phi}_0 , \quad (1.2)$$

with subscripts denoting partial derivatives where convenient. Einstein’s field equations with the electromagnetic field here as source read [3]:

$$U_{uv} = U_u U_v , \quad (1.3)$$

$$2 U_{uu} = U_u^2 + V_u^2 - 2 U_u M_u + 4 \phi_2^2 , \quad (1.4)$$

$$2 U_{vv} = U_v^2 + V_v^2 - 2 U_v M_v + 4 \phi_0^2 , \quad (1.5)$$

$$2 V_{uv} = U_u V_v + U_v V_u + 4 \phi_0 \phi_2 , \quad (1.6)$$

$$2 M_{uv} = V_u V_v - U_u U_v . \quad (1.7)$$

Consider now the head–on collision of a plane light–like shell of matter labelled by a real parameter k (in the sense that if $k = 0$ then the light–like shell is removed) with an electromagnetic shock wave with amplitude b and having a Heaviside step–function profile. We take the history of the signal labelled by k to be found in the region $u > 0, v < 0$ of the space–time with line–element (1.1) while the history of the signal labelled by b is found in the

region $u < 0, v > 0$ of the space-time (the region $u < 0, v < 0$ is taken to be flat so that the signals are non-interacting before collision). To solve the field equations above in the region $u > 0, v > 0$ of the space-time after the collision requires the following conditions on the boundary of this region of space-time: for $u > 0, v = 0$ we require

$$e^{-U} = (1 - k u)^2, \quad V = 0, \quad M = 0, \quad \phi_2 = 0, \quad (1.8)$$

and for $v > 0, u = 0$ we require

$$e^{-U} = \frac{1}{1 + b^2 v^2}, \quad V = 0, \quad e^M = 1 + b^2 v^2, \quad \phi_0 = \frac{b}{1 + b^2 v^2}. \quad (1.9)$$

The solution is given by [1]

$$e^{-U} = (1 - k u)^2 + \frac{1}{1 + b^2 v^2} - 1, \quad (1.10)$$

$$e^{-M - \frac{1}{2}U} = \frac{1 - k u}{(1 + b^2 v^2)^{3/2}}, \quad (1.11)$$

$$\hat{\phi}_0 = \frac{b}{(1 + b^2 v^2)^{3/2}}, \quad (1.12)$$

together with $V = 0$ and $\phi_2 = 0$. The Newman–Penrose components of the Weyl tensor calculated with the metric tensor given via (1.1) are

$$\Psi_0 = -\frac{1}{2}(V_{vv} - U_v V_v + M_v V_v), \quad (1.13)$$

$$\Psi_1 = 0, \quad (1.14)$$

$$\Psi_2 = \frac{1}{4}(V_u V_v - U_u U_v), \quad (1.15)$$

$$\Psi_3 = 0, \quad (1.16)$$

$$\Psi_4 = -\frac{1}{2}(V_{uu} - U_u V_u + M_u V_u). \quad (1.17)$$

Clearly all of these components with the exception of Ψ_2 vanish for the solution given immediately above (because $V = 0$). This is a Petrov Type D Weyl tensor and since Ψ_0 and Ψ_4 vanish there is no backscattered gravitational radiation present. *The purpose of this paper is to establish the existence of backscattered gravitational waves if the light-like shell here is accompanied by an electromagnetic shock wave.* To achieve this it is sufficient to consider a small amplitude accompanying electromagnetic shock wave as a perturbation of the space-time with the functions U, M, ϕ_0 given by (1.10)–(1.12) with $V = 0$ and $\phi_2 = 0$. Without making the small amplitude assumption the collision problem posed here appears intractable (it is a non-trivial

problem even with the small amplitude assumption, as will appear below) notwithstanding the fact that from a physical point of view it is very clear and simple.

2 The Perturbed Space-Time

If the light-like shell above is accompanied by a homogeneous plane electromagnetic shock wave with a step-function profile and amplitude labelled by a then the boundary conditions (1.8) must be replaced by: for $u > 0, v = 0$ we require

$$e^{-U} = \frac{(1 - k u)^2}{1 + a^2 u^2} , \quad V = 0 , \quad e^M = 1 + a^2 u^2 , \quad \phi_2 = \frac{a}{1 + a^2 u^2} . \quad (2.1)$$

Taking the amplitude a to be small we shall neglect squares and higher powers of a and thus replace (2.1) by: for $u > 0, v = 0$ we require

$$e^{-U} = (1 - k u)^2 , \quad V = 0 , \quad M = 0 , \quad \phi_2 = a . \quad (2.2)$$

The field equations and the boundary conditions (1.9) and (2.2) give the following, which are useful later (in particular for determining the boundary conditions to be satisfied by the function \mathcal{K} in (2.19)): for $u > 0, v = 0$:

$$V_u = 0 , \quad V_v = \frac{2 a b u}{1 - k u} , \quad \phi_0 = \frac{b}{1 - k u} , \quad (2.3)$$

and for $v > 0, u = 0$:

$$V_u = 2 a b v , \quad V_v = 0 , \quad \phi_2 = a . \quad (2.4)$$

It is convenient to write V in the form

$$V = \log \left(\frac{1 + A}{1 - A} \right) , \quad (2.5)$$

with $A(u, v)$ small of order a (which we write as $A = O(a)$). Thus we have

$$V_u = 2 A_u = O(a) , \quad V_v = 2 A_v = O(a) . \quad (2.6)$$

We assume in addition that $\phi_2 = O(a)$. Neglecting $O(a^2)$ -terms we have henceforth

$$e^{-U} = (1 - k u)^2 + \frac{1}{1 + b^2 v^2} - 1 . \quad (2.7)$$

Equations (1.3) and (1.7) give, neglecting $O(a^2)$ -terms,

$$(M + \frac{1}{2}U)_{uv} = 0 \Leftrightarrow e^{-M-\frac{1}{2}U} = \frac{1 - k u}{(1 + b^2 v^2)^{3/2}} , \quad (2.8)$$

using the boundary conditions (1.9) and (2.2). Now the first of Maxwell's equations in (1.2) with $O(a^2)$ -terms neglected yields

$$\frac{\partial \hat{\phi}_0}{\partial u} = 0 , \quad (2.9)$$

and solving this using the boundary conditions (1.9) results in

$$\hat{\phi}_0 = \frac{b}{(1 + b^2 v^2)^{3/2}} , \quad (2.10)$$

neglecting $O(a^2)$ -terms. Now (1.4) and (1.5) simplify, neglecting $O(a^2)$ -terms, to

$$2U_{uu} = U_u^2 - 2U_u M_u , \quad (2.11)$$

$$2U_{vv} = U_v^2 - 2U_v M_v + 4\phi_0^2 , \quad (2.12)$$

respectively. These are automatically satisfied by U, M, ϕ_0 given by (2.7), (2.8) and (2.10). The remaining field equations, the second of (1.2) along with (1.6), will determine the two remaining unknown functions V (or equivalently A) and $\hat{\phi}_2$.

To obtain V we begin by eliminating ϕ_2 from (1.6). We start by writing (1.6) in the form

$$2e^{-U}V_{uv} = -\left(e^{-U}\right)_u V_v - \left(e^{-U}\right)_v V_u + 4\hat{\phi}_0 \hat{\phi}_2 . \quad (2.13)$$

Differentiating this with respect to v and multiplying by $\hat{\phi}_0$ yields

$$\begin{aligned} & \hat{\phi}_0 \left\{ 2e^{-U}V_{uvv} + 3\left(e^{-U}\right)_v V_{uv} + \left(e^{-U}\right)_u V_{vv} + \left(e^{-U}\right)_{vv} V_u \right\} \\ &= -2\hat{\phi}_0^3 V_u + \frac{\partial \hat{\phi}_0}{\partial v} \left\{ 2e^{-U}V_{uv} + \left(e^{-U}\right)_u V_v + \left(e^{-U}\right)_v V_u \right\} . \end{aligned} \quad (2.14)$$

With U and $\hat{\phi}_0$ given by (2.7) and (2.10) respectively we find that

$$\hat{\phi}_0 \left(e^{-U}\right)_{vv} + 2\hat{\phi}_0^3 - \left(e^{-U}\right)_v \frac{\partial \hat{\phi}_0}{\partial v} = 0 , \quad (2.15)$$

and so the three terms in (2.14) having V_u as a factor disappear and (2.14) becomes an equation for $W = V_v$ given by

$$\begin{aligned} \left\{ (1 - k u)^2 (1 + b^2 v^2) - b^2 v^2 \right\} W_{uv} &= -3 b^2 v \left\{ (1 - k u)^2 - 1 \right\} W_u \\ &+ k (1 - k u) (1 + b^2 v^2) W_v + 3 k b^2 v (1 - k u) W . \end{aligned} \quad (2.16)$$

To solve this equation it is useful to change the dependent variable W to a new variable $\mathcal{K}(u, v)$ (say) in such a way that the resulting differential equation for \mathcal{K} does not have an undifferentiated \mathcal{K} -term. Multiplying (2.16) by v and writing

$$W = \frac{\mathcal{K}_v}{1 - k u} , \quad (2.17)$$

for some \mathcal{K} , the equation (2.16) implies

$$\left\{ (1 - k u) (1 + b^2 v^2) - \frac{b^2 v^2}{1 - k u} \right\} W_u - k (1 + b^2 v^2) W = \mathcal{K}_u . \quad (2.18)$$

Substituting for W from (2.17) into (2.18) results in

$$\varphi \mathcal{K}_{uv} = \frac{b^2 v^3 k}{(1 - k u)^3} \mathcal{K}_v + \mathcal{K}_u , \quad (2.19)$$

where

$$\varphi = v + b^2 v^3 \left(1 - \frac{1}{(1 - k u)^2} \right) . \quad (2.20)$$

The important difference between this equation and (2.16) is that in this equation the dependent variable \mathcal{K} does not appear undifferentiated (whereas an undifferentiated W appears in (2.16)). This makes it easier to solve (2.19) than (2.16).

3 Construction of a Candidate Solution \mathcal{K}

We look for the solution \mathcal{K} of (2.19) satisfying the following boundary conditions (which follow from (2.3), (2.4), (2.17) and (2.18)) : When $u = 0$ we must have $\mathcal{K}_u = 2 a b v$ and $\mathcal{K}_v = 0$ and when $v = 0$ we require $\mathcal{K}_u = 0$ and $\mathcal{K}_v = 2 a b u$. We will henceforth drop the factor $2 a b$ for the moment and reinstate it at the very end.

First we rewrite (2.19) to read

$$v \mathcal{K}_{uv} - \mathcal{K}_u = b^2 v^3 \left\{ \frac{k}{(1 - k u)^3} \mathcal{K}_v - \left(1 - \frac{1}{(1 - k u)^2} \right) \mathcal{K}_{uv} \right\} . \quad (3.1)$$

This, and the boundary conditions, suggest that we look for a solution which is a power series in powers of b^2 of the form

$$\mathcal{K} = u v + b^2 \mathcal{K}^{(1)} + b^4 \mathcal{K}^{(2)} + \dots \quad (3.2)$$

The first term here is reminiscent of a corresponding term in the Bell–Szekeres [6] solution of the Einstein–Maxwell vacuum field equations. The precise connection with the Bell–Szekeres solution is mentioned in section 6 below. Substitution of (3.2) into (3.1) and equating powers of b^2 on both sides leads to a sequence of differential equations for $\mathcal{K}^{(1)}, \mathcal{K}^{(2)}, \dots$. The boundary conditions above result in the first derivatives of these functions having to vanish when $u = 0$ and when $v = 0$. This will fix the functions uniquely up to a constant in each case. We can take this constant to vanish because ultimately it can be removed by rescaling the coordinates x and y in (1.1). The differential equations for the coefficients in (3.2) read

$$v \mathcal{K}_{uv}^{(1)} - \mathcal{K}_u^{(1)} = -v^3 + \frac{v^3}{(1 - k u)^3}, \quad (3.3)$$

$$v \mathcal{K}_{uv}^{(i)} - \mathcal{K}_u^{(i)} = v^3 \left\{ \frac{k}{(1 - k u)^3} \mathcal{K}_v^{(i-1)} - \left(1 - \frac{1}{(1 - k u)^2} \right) \mathcal{K}_{uv}^{(i-1)} \right\} \quad (3.4)$$

for $i = 2, 3, 4, \dots$. These equations are straightforward to solve subject to the boundary conditions given above. The first few functions are given by

$$\mathcal{K}^{(1)} = \frac{v^3}{2} \left\{ -u + \frac{1}{2k(1 - k u)^2} \right\}, \quad (3.5)$$

$$\mathcal{K}^{(2)} = \frac{3v^5}{8} \left\{ u - \frac{1}{k(1 - k u)^2} + \frac{3}{8k(1 - k u)^4} \right\}, \quad (3.6)$$

$$\mathcal{K}^{(3)} = \frac{5v^7}{16} \left\{ -u + \frac{3}{2k(1 - k u)^2} - \frac{9}{8k(1 - k u)^4} + \frac{5}{16k(1 - k u)^6} \right\}. \quad (3.7)$$

Substituting into (3.2) we now have

$$\begin{aligned} \mathcal{K} = & u v - \frac{1}{2} b^2 v^3 u + \frac{3}{8} b^4 v^5 u - \frac{5}{16} b^6 v^7 u + \dots \\ & - \frac{1}{2} \frac{b^2 v^3}{k} \left(-\frac{1}{2} \chi \right) + \frac{3}{8} \frac{b^4 v^5}{k} \left(-\frac{1}{2} \chi + \frac{3}{8} \chi^2 \right) \\ & - \frac{5}{16} \frac{b^6 v^7}{k} \left(-\frac{1}{2} \chi + \frac{3}{8} \chi^2 - \frac{5}{16} \chi^3 \right) + \dots, \end{aligned} \quad (3.8)$$

where the variable

$$\chi = \frac{1}{(1 - k u)^2} - 1, \quad (3.9)$$

has been introduced for convenience. This variable (which vanishes when $u = 0$) will appear frequently in the sequel. Remembering that we are here constructing *a candidate* exact solution of (2.19) satisfying the boundary conditions (the candidate will be verified to be an exact solution in the next section), the form of (3.8) suggests we should write

$$\begin{aligned} \mathcal{K} = & \frac{u v}{\sqrt{1 + b^2 v^2}} \\ & - \frac{1}{2} \frac{b^2 v^3}{k} \left(1 - \frac{1}{2} \chi\right) + \frac{3}{8} \frac{b^4 v^5}{k} \left(1 - \frac{1}{2} \chi + \frac{3}{8} \chi^2\right) \\ & - \frac{5}{16} \frac{b^6 v^7}{k} \left(1 - \frac{1}{2} \chi + \frac{3}{8} \chi^2 - \frac{5}{16} \chi^3\right) + \dots \\ & + \left[\frac{1}{2} \frac{b^2 v^3}{k} - \frac{3}{8} \frac{b^4 v^5}{k} + \frac{5}{16} \frac{b^6 v^7}{k} - \dots \right] . \end{aligned} \quad (3.10)$$

The final series in square brackets here can tentatively be written

$$- \frac{v}{k} \left\{ \frac{1}{\sqrt{1 + b^2 v^2}} - 1 \right\} . \quad (3.11)$$

Putting this into (3.10) results in

$$\mathcal{K} = - \frac{v(1 - k u)}{k \sqrt{1 + b^2 v^2}} + \frac{v}{k} \mathcal{F} , \quad (3.12)$$

with

$$\begin{aligned} \mathcal{F} = & 1 - \frac{1}{2} b^2 v^2 \left(1 - \frac{1}{2} \chi\right) + \frac{3}{8} b^4 v^4 \left(1 - \frac{1}{2} \chi + \frac{3}{8} \chi^2\right) \\ & - \frac{5}{16} b^6 v^6 \left(1 - \frac{1}{2} \chi + \frac{3}{8} \chi^2 - \frac{5}{16} \chi^3\right) + \dots . \end{aligned} \quad (3.13)$$

From this we calculate that

$$\begin{aligned} \mathcal{F} + 2(\chi + 1) \frac{\partial \mathcal{F}}{\partial \chi} = & 1 + 3 \left(\frac{1}{2}\right)^2 b^2 v^2 \chi + 5 \left(\frac{3}{8}\right)^2 b^4 v^4 \chi^2 \\ & + 7 \left(\frac{5}{16}\right)^2 b^6 v^6 \chi^3 + 9 \left(\frac{35}{128}\right)^2 b^8 v^8 \chi^4 + \dots \end{aligned} \quad (3.14)$$

We can rewrite this in the form

$$\mathcal{F} + 2(\chi + 1) \frac{\partial \mathcal{F}}{\partial \chi} = \frac{\partial}{\partial v} (v \mathcal{G}) , \quad (3.15)$$

with

$$\begin{aligned}\mathcal{G} = & 1 + \left(\frac{1}{2}\right)^2 b^2 v^2 \chi + \left(\frac{3}{8}\right)^2 b^4 v^4 \chi^2 \\ & + \left(\frac{5}{16}\right)^2 b^6 v^6 \chi^3 + \left(\frac{35}{128}\right)^2 b^8 v^8 \chi^4 + \dots\end{aligned}\quad (3.16)$$

This suggests that we should write

$$\mathcal{G} = \sum_{n=0}^{\infty} \left[\frac{(2n)!}{2^{2n}(n!)^2} \right]^2 (y\chi)^n = \frac{2}{\pi} K(y\chi), \quad (3.17)$$

where $y = b^2 v^2$ and K is the complete elliptic integral of the first kind (see Eq.(A.1)) with argument $y\chi$. Substituting this into (3.15) we can then integrate with respect to χ to obtain

$$\mathcal{F} = \frac{2}{\pi} \Pi(-y, y\chi) + c_0(y), \quad (3.18)$$

where Π is the complete elliptic integral of the third kind (see Eq.(A.3)) and c_0 is a function of integration. The boundary conditions are satisfied provided $\mathcal{F} = (1+y)^{-1/2}$ when $\chi = 0 \Leftrightarrow u = 0$. This is true of (3.18) provided the function of integration $c_0(y) = 0$. Now (3.12) multiplied by the factor $2ab$, and with \mathcal{F} given by (3.18) with $c_0 = 0$, is a candidate exact solution of (2.19) satisfying the required boundary conditions.

4 Verifying the Candidate Solution \mathcal{K}

The candidate solution constructed in the previous section reads:

$$\mathcal{K} = -\frac{2abv(1-ku)}{k\sqrt{1+b^2v^2}} + \frac{4ab}{\pi k} v \Pi(-b^2v^2, b^2v^2\chi). \quad (4.1)$$

It is easy to check that the first term satisfies the differential equation (2.19). Hence we wish to demonstrate that

$$\tilde{\mathcal{K}} = v \Pi(-b^2v^2, b^2v^2\chi), \quad (4.2)$$

satisfies (2.19). Writing $n = -b^2v^2$ and $m = b^2v^2\chi$ and substituting (4.2) into (2.19) results in $\Pi(n, m)$ having to satisfy the differential equation

$$2n(m-1) \frac{\partial^2 \Pi}{\partial n \partial m} + 2m(m-1) \frac{\partial^2 \Pi}{\partial m^2} + n \frac{\partial \Pi}{\partial n} + 2(2m-1) \frac{\partial \Pi}{\partial m} + \frac{1}{2} \Pi = 0. \quad (4.3)$$

Using the formula (A.6) for the partial derivative of Π with respect to m this can be rewritten in the form

$$\begin{aligned} n(n-1) \frac{\partial \Pi}{\partial n} &+ (-3m^2 + m + 4mn - 2n) \frac{\partial \Pi}{\partial m} + \frac{1}{2}(-3m + n + 2) \Pi \\ &+ \frac{(1-3m)}{2(m-1)} E - \frac{1}{2} K = 0, \end{aligned} \quad (4.4)$$

where K and E are the complete elliptic integrals of the first and second kind respectively (see Appendix). The formulas giving the partial derivatives of Π with respect to m and n in terms of the complete elliptic integrals are given in (A.6) and (A.7) respectively. When these substitutions are made in (4.4) it follows that (4.4) is an identity. Hence the candidate solution (4.1) of (2.19) is indeed an exact solution of (2.19).

5 Perturbed Fields

In order to demonstrate explicitly that the perturbed field we are constructing here contains backscattered gravitational radiation we must use the function \mathcal{K} we have obtained to calculate the leading terms (for small a) in Ψ_0 and Ψ_4 given by (1.13) and (1.17) and see that they are non-vanishing. This we can now do because with (2.5) and (2.17), with $W = V_v$, we have

$$V = 2A = \frac{\mathcal{K}}{(1 - ku)}, \quad (5.1)$$

neglecting $O(a^2)$ -terms, with \mathcal{K} given by (4.1). In integrating (2.17) with respect to v a function of u of integration has been put equal to zero on account of the fact that by (2.2) V must vanish when $v = 0$ and we have already ensured that \mathcal{K} vanishes when $v = 0$. Since V is small of order a we can enter the ‘background’ expressions (2.7) and (2.8) for U and M in (1.13). This initially results in

$$\Psi_0 = -\frac{1}{2}V_{vv} - \frac{3b^2v}{2(1+b^2v^2)}V_v + \frac{3}{4}U_vV_v. \quad (5.2)$$

Introducing again $n = -b^2v^2$ and $m = b^2v^2\chi$ we obtain from (2.7) the expression

$$U_v = -\frac{2(n-m)}{v(n-1)(m-1)}. \quad (5.3)$$

By (5.1) with \mathcal{K} given by (4.1) we obtain, using (A.6) and (A.7),

$$V_v = -\frac{2ab}{k(1-n)^{3/2}} + \frac{4ab}{\pi k(1-ku)(n-1)} \left\{ -\Pi + K + \frac{1}{m-1}E \right\}, \quad (5.4)$$

and

$$\frac{1}{2}V_{vv} + \frac{3b^2v}{2(1+b^2v^2)}V_v = -\frac{2ab}{\pi kv(1-ku)(n-1)(m-1)} \left\{ K + \frac{(m+1)}{(m-1)}E \right\}, \quad (5.5)$$

with Π a function of n, m and K, E functions of m . Thus Ψ_0 in (5.2) has the non-zero value

$$\begin{aligned} \Psi_0 = & \frac{2ab}{\pi kv(1-ku)(n-1)^2(m-1)} \{ 3(n-m)\Pi - (2n-3m+1)K \\ & - \frac{(2n-2m-nm+1)}{(m-1)}E \} - \frac{3ab(n-m)}{kv(1-n)^{5/2}(m-1)}, \end{aligned} \quad (5.6)$$

which indicates that following the collision of the light-like shell, accompanied by the small amplitude electromagnetic waves labelled by a , with the electromagnetic waves labelled by b there exist backscattered gravitational waves having propagation direction in space-time $\partial/\partial u$. In similar fashion we find that

$$V_u = -\frac{4abv}{\pi(1-ku)^2(m-1)}E, \quad (5.7)$$

and

$$\Psi_4 = -\frac{2abkv}{\pi\chi(1-ku)^3(m-1)} \left\{ K - \frac{(2m-1)}{(m-1)}E \right\}. \quad (5.8)$$

The non-vanishing of this quantity indicates the existence of backscattered gravitational radiation after the collision having propagation direction in space-time $\partial/\partial v$. Since V is small of first order and, neglecting $O(a^2)$ -terms, U is given by (2.7), we see that Ψ_2 in (1.15) has its background value in the linear approximation given by

$$\Psi_2 = \frac{kn}{v(1-ku)^3(m-1)^2}. \quad (5.9)$$

To discover what type of physical signal has as its history in space-time the boundary $u > 0, v = 0$ or $v > 0, u = 0$ of the interaction region of the space-time we carry out the calculation above replacing u with $u_+ = u\vartheta(u)$ and v with $v_+ = v\vartheta(v)$ where ϑ is the Heaviside step-function. We find that

in addition to the backscattered gravitational radiation found above there are Dirac delta-function terms in Ψ_0 and Ψ_4 given by

$$\delta\Psi_0 = -\frac{a b u_+}{(1 - k u_+)} \delta(v) , \quad \delta\Psi_4 = -a b v_+ \delta(u) , \quad (5.10)$$

indicating that the boundaries of the interaction region are the histories of impulsive gravitational waves. We note that the Ricci tensor possesses a delta function term, $\delta R_{ij} = -2k\delta(u)u_{,i}u_{,j}$, reflecting the presence of the light-like shell (labelled by k) having history $u = 0$.

The Maxwell field after the collision has two radiative components conveniently described by $\hat{\phi}_0$ and $\hat{\phi}_2$. In the approximation in which $O(a^2)$ -terms are neglected we have already seen that $\hat{\phi}_0$ is given by (2.10). With \mathcal{K} and thus V already known in this approximation we turn to the second of the Maxwell equations (1.2) to find $\hat{\phi}_2$. Neglecting $O(a^2)$ -terms this equation reads

$$\frac{\partial \hat{\phi}_2}{\partial v} = -\frac{b}{2(1 + b^2 v^2)^{3/2}} \frac{\partial}{\partial u} \left(\frac{\mathcal{K}}{(1 - k u)} \right) . \quad (5.11)$$

Using (4.1) and (A.3) with, for convenience, the substitution $b^2 v^2 = y$ this simplifies to

$$\frac{\partial \hat{\phi}_2}{\partial y} = -\frac{a}{\pi(1 - k u)^2(1 + y)^{3/2}} \int_0^{\pi/2} \frac{d\theta}{(1 - y \chi \sin^2 \theta)^{3/2}} . \quad (5.12)$$

Noting that

$$\frac{d}{dy} \left(\frac{1 - \chi(1 + 2y) \sin^2 \theta}{\sqrt{1 + y} \sqrt{1 - y \chi \sin^2 \theta}} \right) = -\frac{(1 + \chi \sin^2 \theta)^2}{2(1 + y)^{3/2}(1 - y \chi \sin^2 \theta)^{3/2}} , \quad (5.13)$$

we can integrate (5.12) to obtain

$$\hat{\phi}_2 = \frac{2a}{\pi(1 - k u)^2 \sqrt{1 + y}} \int_0^{\pi/2} \frac{(1 - \chi(1 + 2y) \sin^2 \theta) d\theta}{(1 + \chi \sin^2 \theta)^2 \sqrt{1 - y \chi \sin^2 \theta}} . \quad (5.14)$$

There is no need for a function of χ of integration since evaluating this at $v = 0$ (which corresponds to $y = 0$) yields the correct boundary value of $\hat{\phi}_2 = a(1 - k u)$ (see Eq.(2.2)). Rearranging the integrand in (5.14) permits us to write

$$\begin{aligned} \hat{\phi}_2 &= -\frac{4a\chi\sqrt{1+y}}{\pi(1 - k u)^2} \int_0^{\pi/2} \frac{\sin^2 \theta d\theta}{(1 + \chi \sin^2 \theta)^2 \sqrt{1 - y \chi \sin^2 \theta}} \\ &\quad + \frac{2a}{\pi(1 - k u)^2 \sqrt{1 + y}} \Pi(-\chi, y\chi) , \end{aligned} \quad (5.15)$$

with $\Pi(n, m)$ given by (A.3). Making use of (A.7) we have

$$\int_0^{\pi/2} \frac{\sin^2 \theta d\theta}{(1 + \chi \sin^2 \theta)^2 \sqrt{1 - y \chi \sin^2 \theta}} = \frac{(\chi - y) \Pi(-\chi, y \chi)}{2 \chi (\chi + 1) (y + 1)} + \frac{K(y \chi)}{2 \chi (\chi + 1)} - \frac{E(y \chi)}{2 \chi (\chi + 1) (y + 1)}, \quad (5.16)$$

and this simplifies (5.15) to read

$$\hat{\phi}_2 = \frac{2a}{\pi} \sqrt{1 + y} \{ \Pi(-\chi, y \chi) - K(y \chi) \} + \frac{2a}{\pi} \frac{E(y \chi)}{\sqrt{1 + y}}. \quad (5.17)$$

Due to the presence of $\hat{\phi}_0$ and $\hat{\phi}_2$ after the collision, two systems of backscattered electromagnetic radiation exist having propagation directions $\partial/\partial u$ and $\partial/\partial v$ in the interaction space-time.

6 Discussion

Neglecting $O(a^2)$ -terms we have obtained explicit expressions for the functions U, M, V in the line-element (1.1). These are given respectively by (2.7), (2.8) and by (5.1) with (4.1). We also have explicit expressions for the Maxwell field described by the functions $\hat{\phi}_0$ and $\hat{\phi}_2$. These functions are found in (2.10) and (5.17) respectively.

The approximate solution, for small a , of the Einstein-Maxwell field equations described above has a limit $k \rightarrow 0$ corresponding to the removal of the light-like shell. In this limit (4.1) gives $\mathcal{K} \rightarrow 2abuv$ while $\Psi_0 \rightarrow 0$ by (5.6), $\Psi_4 \rightarrow 0$ by (5.8) and $\Psi_2 \rightarrow 0$ by (5.9) so that the backscattered gravitational waves disappear, but the impulsive gravitational waves (5.10) remain with $k = 0$ in ${}^\delta\Psi_0$. All of this corresponds to the important Bell-Szekeres [6] solution giving the exact conformally flat space-time following the collision of the two electromagnetic shock waves labelled by a and b . As a converse to the problem considered in this paper one might ask whether the introduction of a light-like shell accompanying one of these waves would produce backscattered gravitational radiation after collision. It is sufficient to answer this question in the affirmative by assuming small a and small k , where k is proportional to the energy density of the light-like shell measured by specified observers [2], and to specialize Ψ_0 and Ψ_4 , given by (5.6) and (5.8) respectively, to this case and see that they are non-zero. We find that

$$\Psi_0 = ab^3k u^2v \left\{ \frac{9(2 + 7b^2v^2 + 4b^4v^4)}{(1 + b^2v^2)^{5/2}} + \frac{7 + 7b^2v^2 + 3b^4v^4}{2(1 + b^2v^2)^2} \right\}, \quad (6.1)$$

$$\Psi_4 = \frac{3}{2} a b^3 k v^3 , \quad (6.2)$$

neglecting $O(k^2)$ -terms. In addition (5.9) becomes, for small k ,

$$\Psi_2 = -b^2 k v + O(k^2) , \quad (6.3)$$

and these last three formulas confirm the limits mentioned above.

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A Formulas for Complete Elliptic Integrals

The complete elliptic integrals of the first, second and third kinds are given respectively by [7]

$$K = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-m \sin^2 \theta}} = K(m) , \quad (\text{A.1})$$

$$E = \int_0^{\pi/2} \sqrt{1-m \sin^2 \theta} d\theta = E(m) , \quad (\text{A.2})$$

$$\Pi = \int_0^{\pi/2} \frac{d\theta}{(1-n \sin^2 \theta) \sqrt{1-m \sin^2 \theta}} = \Pi(n, m) , \quad (\text{A.3})$$

where m, n are real constants. The well-known formulas for their derivatives are

$$\frac{dK}{dm} = -\frac{1}{2m} \left(K + \frac{1}{m-1} E \right) , \quad (\text{A.4})$$

$$\frac{dE}{dm} = \frac{1}{2m} (E - K) , \quad (\text{A.5})$$

$$\frac{\partial \Pi}{\partial m} = \frac{1}{2(n-m)} \left\{ \Pi + \frac{1}{m-1} E \right\} , \quad (\text{A.6})$$

$$\frac{\partial \Pi}{\partial n} = -\frac{1}{2(n-m)(n-1)} \left\{ \frac{(n^2-m)}{n} \Pi - \frac{(n-m)}{n} K + E \right\} . \quad (\text{A.7})$$